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Some Exact Solutions to the Sine-Gordon Equations

Z. J. Yang[1,2]

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We present some new exact (solitary and traveling) solutions to the sine-Gordon equations $c^2u_{xx} - u_{tt} = \alpha \sin u$ and $cu_{xt} + u_{tt} = \alpha \sin u$, and the dissipative sine-Gordon equation $c^2u_{xx} - u_{tt} - \gamma u_t = \alpha_1 \sin u + \alpha_2 \sin(2u)$. Here, α, α_1, and α_2 are the coupling constants, real numbers; c is the *intrinsic* speed of the propagating wave determined by the system; and γ is the damping factor. The physical implication is briefly discussed.

In this paper, we report some new exact (solitary and traveling) solutions to the sine-Gordon equations and the modified/dissipative sine-Gordon equation, introducing a damping term and a higher-order approximation.

I. Let us consider the sine-Gordon equation

$$c^2u_{xx} - u_{tt} = \alpha \sin u \tag{1}$$

which has been widely used in the physical sciences (Barone *et al.*, 1971; Scott *et al.*, 1973; Lamb, 1971; Drazin and Johnson, 1989; Sachdev, 1987; Kivshar and Malomed, 1989; Mikeska and Steiner, 1991; Newell and Moloney, 1992). Here α is the *coupling constant*, a real number; c is the *intrinsic* speed of the propagating wave determined by the system. A well-known exact (solitary and traveling) solution is

$$u(x - vt) = 4 \arctan\left\{\exp\left[\pm\left(\frac{\alpha}{c^2 - v^2}\right)^{1/2} (x - vt + c_0)\right]\right\} \tag{2}$$

where v is a positive real number defined as the speed of the imposed traveling wave, and c_0 is an arbitrary constant determined by the initial condition.

[1] Department of Physics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5.
[2] Present address: ET/335, Argonne National Laboratory, Argonne, Illinois 60439.

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In this paper, we present a new exact (solitary and traveling) solution

$$u(x - vt) = \pm 2 \arctan\left\{\sinh\left[\left(\frac{\alpha}{v^2 - c^2}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{3}$$

To our best knowledge, this solution has not been reported in the literature.

As far as traveling solutions are concerned, we use the variable transformation

$$\xi = x - vt \tag{4}$$

Thus, equation (1) is simplified to be

$$(c^2 - v^2)u'' = \alpha \sin u \tag{5}$$

Let us consider the following *ansatz*:

$$\frac{du}{d\xi} = u' = ab \cos \frac{u}{a} \tag{6}$$

where a and b are undetermined real numbers.

Direct integration of equation (6) yields the solution

$$u(x - vt) = u(\xi) = a \arctan\{\sinh[b(\xi + c_0)]\} \tag{7}$$

From equation (6), we have

$$u'' = -\frac{ab^2}{2} \sin \frac{2u}{a} \tag{8}$$

Substituting this equation into equation (5) yields the equality

$$-\frac{1}{2} ab^2(c^2 - v^2)\sin \frac{2u}{a} \equiv \alpha \sin u \tag{9}$$

Let $a = 2$; we obtain

$$b = \pm\left(\frac{\alpha}{v^2 - c^2}\right)^{1/2} \tag{10}$$

Equation (3) is obtained by substituting these results into equation (7).

The mathematical differences between equations (2) and (3) can be easily seen. Due to the failure of the superposition principle for nonlinear systems, equation (3) cannot be derived from equation (2) by a simple algebra. It is straightforward to show that equation (3) cannot be obtained by the Bäcklund transformation either. Furthermore, in order for the solutions (2) and (3) to be real functions, when $\alpha > 0$, $v < c$ for equation (2) and $v > c$ for equation (3). On the other hand, when $\alpha < 0$, $v > c$ for equation (2) and

$v < c$ for equation (3). This point will be stressed from a physical point of view below.

Now, let us see the differences in physics between equations (2) and (3). In the field theory, only $\alpha > 0$ is allowed, and c is the speed of the light in vacuum, which is the upper limit of any observable physical propagations. Thus, equation (3) may not be used to describe an observable physical parameter. However, in other systems, such as optical pulse propagation in a resonant medium (Lamb, 1971) and one-dimensional magnets (Mikeska and Steiner, 1991), etc., $\alpha < 0$ is allowed. Furthermore, in these systems, when $\alpha < 0$, the solution for $v < c$, equation (2), may not correspond to an observable physical property either. Therefore, if both equations (2) and (3) describe two measurable physical processes in a system, the two events are substantially different. A Cherenkov effect-like mechanism may be considered as a heuristic argument. For $\alpha > 0$, equation (2) describes the normal solitary propagation, while equation (3) expresses the Cherenkov-like propagation whose instability (Barone *et al.*, 1971) corresponds to an energy absorption/radiation. The parallel conclusions should also be true: for $\alpha < 0$, equation (2) describes the Cherenkov-like propagation, while equation (3) expresses the normal solitary propagation. This argument needs to be verified by experiments.

In addition, it is easy to show that equation (3) has other identical representations as

$$u(x - vt) = \pm 2 \arcsin\left\{\tanh\left[\left(\frac{\alpha}{v^2 - c^2}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{11}$$

and

$$u(x - vt) = \pm 4 \arctan\left\{\tanh\left[\frac{1}{2}\left(\frac{\alpha}{v^2 - c^2}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{12}$$

II. Let us consider the sine-Gordon equation

$$cu_{xt} + u_{tt} = \alpha \sin u \tag{13}$$

which is used to describe the propagation of an ultrashort laser pulse in a medium whose absorption bands are near or at the frequency of an applied pulse (Barone *et al.*, 1971).

Using the *ansatz*

$$\frac{du}{d\xi} = u' = \frac{ab}{2} \sin \frac{2u}{a} \tag{14}$$

we can easily obtain the solution as

$$u(x - vt) = 4 \arctan\left\{\exp\left[\pm\left(\frac{\alpha}{v^2 - cv}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{15}$$

The real function of the solution requires $v > c$ for $\alpha > 0$, and $v < c$ for $\alpha < 0$.

Using *ansatz* (6) and equation (8), we obtain the solution

$$u(x - vt) = \pm 2 \arctan\left\{\sinh\left[\left(\frac{\alpha}{cv - v^2}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{16}$$

Similarly, it requires $v < c$ for $\alpha > 0$, and $v > c$ for $\alpha < 0$.

It is trivial to show that equation (16) has other identical representations as

$$v(x - vt) = \pm 2 \arcsin\left\{\tanh\left[\left(\frac{\alpha}{cv - v^2}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{17}$$

and

$$u(x - vt) = \pm 4 \arctan\left\{\tanh\left[\frac{1}{2}\left(\frac{\alpha}{cv - v^2}\right)^{1/2}(x - vt + c_0)\right]\right\} \tag{18}$$

III. Let us consider the dissipative sine-Gordon equation

$$c^2 u_{xx} - u_{tt} - \gamma u_t = \alpha_1 \sin u + \alpha_2 \sin(2u)$$

Here, the *coupling constants* α_1 and α_2 are real numbers, and the *damping factor* γ is also a real number. The $\alpha_2 \sin(2u)$ term is modeled as a higher-order approximation compared with the origin sine-Gordon equation. It is obvious that the introduction of the two new terms makes the equation more realistic for describing the physical systems.

Through introducing the *ansatz* (14), we previously obtained an exact traveling (solitary kink) solution as (Yang, 1994; Yang *et al.*, 1994)

$$u(x - vt) = 2 \arctan\left[\exp\left(\pm\frac{(2\alpha_2\gamma^2 + \alpha_1^2)^{1/2}}{c\gamma}x - \frac{\alpha_1}{\gamma}t + c_0\right)\right] \tag{19}$$

Using *ansatz* (6) and equation (8), we have obtained another exact traveling solution as

$$u(x - vt) = \frac{\pi}{2} - \arctan\left[\sinh\left(\pm\frac{(2\alpha_2\gamma^2 + \alpha_1^2)^{1/2}}{c\gamma}x + \frac{\alpha_1}{\gamma}t + c_0\right)\right] \tag{20}$$

When $\alpha_2 = 0$, i.e., without considering the $\sin(2u)$ term, the solutions will be easily simplified.

As discussed above, the differences between these two solutions are clear to see. Similarly, the new solution has other identical representations:

$$u(x - vt) = \frac{\pi}{2} - \arcsin\left[\tanh\left(\pm \frac{(2\alpha_2\gamma^2 + \alpha_1^2)^{1/2}}{c\gamma} x + \frac{\alpha_1}{\gamma} t + c_0\right)\right] \tag{21}$$

and

$$u(x - vt) = \frac{\pi}{2} - 2\arctan\left[\tanh\left(\pm \frac{(2\alpha_2\gamma^2 + \alpha_1^2)^{1/2}}{2c\gamma} x + \frac{\alpha_1}{2\gamma} t + c_0\right)\right] \tag{22}$$

In summary, by introducing a proper *ansatz*, we have obtained some exact (solitary and traveling) solutions to the sine-Gordon equations and a modified/dissipative sine-Gordon equation. The corresponding physical interpretations need to be verified by experiments.

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REFERENCES

Barone, A., Esposito, F., Magee, C. J., and Scott, A. C. (1971). *Nuovo Cimento,* **1**, 227.

Drazin, P. G., and Johnson, R. S. (1989). *Solitons: An Introduction,* Cambridge University Press, Cambridge.

Kivshar, Y. S., and Malomed, B. A. (1989). *Review of Modern Physics,* **61**, 763.

Lamb, Jr., G. L. (1971). *Reviews of Modern Physics,* **43**, 99.

Mikeska, H. J., and Steiner, M. (1991). *Advances in Physics,* **40**, 191.

Newell, A. C., and Moloney, J. V. (1992). *Nonlinear Optics,* Addison-Wesley, Redwood City, California.

Sachdev, P. L. (1987). *Nonlinear Diffusive Waves,* Cambridge University Press, Cambridge.

Scott, A. C., Chu, F. Y. F., and McLaughlin, D. W. (1973). *Proceedings IEEE,* **61**, 1443.

Yang, Z. J. (1994). *Journal of Physics A,* **27**, 2837.

Yang, Z. J., Dunlap, R. A., and Geldart, D. J. W. (1994). *International Journal of Theoretical Physics,* **33**, 2057.